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The Landau Inequality for Bounded Intervals with $\|f^{(3)}\|$ Finite

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1. INTRODUCTION AND THE THEOREM

Landau's problem concerning inequalities between derivatives on the whole real line or on the half real line has been discussed in the general settings by many authors; see, e.g., Kolmogorov [4], Cavaretta [1] and Schoenberg and Cavaretta [7]. On the other hand, in 1975, Chui and Smith [2] treated Landau's problem for bounded intervals with $\|f''\|$ finite, and recently Pinkus [5] studied a pointwise version of Landau's problem for bounded intervals. (For a simple proof of the Chui–Smith theorem we refer the reader to [6].) In this paper we intend to discuss the problem for bounded intervals with $\|f^{(3)}\|$ finite. It will be seen that Karlin's conjecture [3, p. 423], which asserts that a Zolotarev spline is extremal with respect to any derivative, is true for the third derivative (it is also true for the second).

To formulate our result, let f be a real function on the unit interval $I = [0, 1]$ such that f'' is absolutely continuous on I; thus $f^{(3)}(t)$ exists for almost all $t \in I$ with respect to Lebesgue measure, and

$$f''(x) - f''(0) = \int_0^x f^{(3)}(t)\,dt \qquad \text{for all} \quad x \in I.$$

Put $\|f\|_I = \operatorname{ess\,sup}_{t\in I} |f(t)|$. Then the result can be formulated as follows:

THEOREM. *Let $\|f\|_I \leqslant 1$ and $\|f^{(3)}\|_I \leqslant A$. Then*

$$\|f'\|_I \leqslant \frac{4}{\alpha} + \frac{A}{6}\alpha^2 \qquad \text{and} \qquad \|f''\|_I \leqslant \frac{4}{\alpha^2} + \frac{2A}{3}\alpha \tag{1}$$

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if $0 \leqslant A < 81$, *where* α *is defined by the relations*

$$\tfrac{1}{3} < \alpha \leqslant \tfrac{1}{2} \qquad \textit{and} \qquad A = 24(\tfrac{1}{2} - \alpha)/\alpha^2(1-\alpha)^2;$$

$$\|f'\|_I \leqslant \frac{9}{2}\sqrt[3]{\frac{A}{3}} \qquad \textit{and} \qquad \|f''\|_I \leqslant 6\left(\sqrt[3]{\frac{A}{3}}\right)^2 \tag{2}$$

if $A \geqslant 81$. *Moreover these inequalities are best possible.*

2. Proof of the Theorem

First of all we shall construct an extremizing function f_0 on I. To do this, let α be the constant introduced in the theorem if $0 \leqslant A < 81$, and put

$$\left(\frac{1}{3} \geqslant\right) \alpha = \sqrt[3]{\frac{3}{A}} \qquad \text{if} \quad A \geqslant 81.$$

We shall first consider the case $A > 0$. Let $a > 0$ be fixed arbitrarily, and put

$$h(t) = \frac{A}{6}(t+a)\,t(t-a) \qquad \text{for all} \quad t \leqslant a/\sqrt{3}.$$

Then, immediately,

$$h'(t) = \frac{A}{2}(t + a/\sqrt{3})(t - a/\sqrt{3}).$$

$$h''(t) = At \qquad \text{and} \qquad h^{(3)}(t) = A.$$

Hence, in particular, $h'(a/\sqrt{3}) = 0$, and thus it may be readily seen that there exists a unique extension g of h to the right such that g'' is absolutely continuous on $[a/\sqrt{3}, \infty)$ and also such that

$$|g(t)| \leqslant -h(a/\sqrt{3}) \qquad \text{for all} \quad t \geqslant a/\sqrt{3}$$

and

$$|g^{(3)}(t)| \leqslant A \qquad \text{for almost all} \quad t \geqslant a/\sqrt{3}.$$

More explicitly, g can be defined by

$$\begin{aligned} g(t) &= h(t) && \text{for} \quad t \leqslant a/\sqrt{3} \\ &= h(2a/\sqrt{3} - t) && \text{for} \quad a/\sqrt{3} < t \leqslant \sqrt{3}\,a \\ &= g(t - 4a/\sqrt{3}) && \text{for} \quad t > \sqrt{3}\,a. \end{aligned}$$

On the other hand, since the function h satisfies

$$h(-2a/\sqrt{3}) = h(a/\sqrt{3}) = -Aa^3/9\sqrt{3},$$

we see that if $a > 0$ is such that $Aa^3/9\sqrt{3} = 1$, then

$$\frac{a}{\sqrt{3}} = \sqrt[3]{\frac{3}{A}} \qquad \text{and} \qquad |g(t)| \leqslant 1 \qquad \text{for all} \quad t \geqslant -2a/\sqrt{3}.$$

Thus, if $A \geqslant 81$ then, recalling that $\alpha = \sqrt[3]{(3/A)} \leqslant 1/3$, an extremizing function f_0 can be defined by

$$f_0(t) = g(t - 2\alpha) \qquad \text{for all} \quad t \in I.$$

If $0 < A < 81$ then $a/\sqrt{3} = \sqrt[3]{(3/A)} > 1/3$; and in this case, the above examination shows that an extremizing function f_0 can be defined by solving the simultaneous equations:

$$f_0(t) = \frac{A}{6}t^3 + Bt^2 + Ct - 1, \qquad f_0(1) = -1,$$

$$f_0(\alpha) = 1, \qquad f_0'(\alpha) = 0 \quad \text{and} \quad 1/3 < \alpha \leqslant 1/2.$$

The answer is as follows:

$$f_0(t) = \frac{A}{6}t^3 - \left(\frac{2}{\alpha^2} + \frac{A}{3}\alpha\right)t^2 + \left(\frac{4}{\alpha} + \frac{A}{6}\alpha^2\right)t - 1 \qquad \text{on} \quad I,$$

where α is such that

$$A = 24(\tfrac{1}{2} - \alpha)/\alpha^2(1 - \alpha)^2.$$

It is easily seen that this function is also an extremizing function for the case $A = 0$. (Here the author would like to remark that she learned from one of the referees that the above polynomial function is simply the associated Zolotarev polynomial. Also, she would like to remark that the quartic equation defining α, when $0 \leqslant A < 81$, can be explicitly solved, but the explicit formula is not given here, because the formula is neither simple nor essential.)

Next we shall verify some properties of f_0 needed in the proof of the theorem. Before doing so, we shall set

$$\beta = \min\{3\alpha, 1\} \qquad \text{and} \qquad \beta^* = \beta + 2k\alpha,$$

where $k \geqslant 0$ is an integer such that $\beta + 2k\alpha \leqslant 1 < \beta + 2(k+1)\alpha$. Then f_0 satisfies the following properties:

(a) $\|f_0\|_I = f_0(\alpha) = 1, f_0(0) = f_0(\beta) = -1$ and $|f(\beta^*)| = 1$.

(b) If $0 \leqslant A < 81$, then

$$\|f'_0\|_I = f'_0(0) = \frac{4}{\alpha} + \frac{A}{6}\alpha^2 \text{ and } \|f''_0\|_I = -f''_0(0) = \frac{4}{\alpha^2} + \frac{2A}{3}\alpha.$$

(c) If $A \geqslant 81$, then

$$\|f'_0\|_I = f'_0(0) = \frac{9}{2}\sqrt[3]{\frac{A}{3}} \qquad \text{and} \qquad \|f''_0\|_I = -f''_0(0) = 6\left(\sqrt[3]{\frac{A}{3}}\right)^2.$$

(d) $f''_0(t) = At - \|f''_0\|_I$ on $[0, \beta]$, and $f''_0(\alpha) < 0$.

In fact, (a) is immediate from the definition of f_0. If $A \geqslant 81$ then, since $\alpha^3 = 3/A$ and

$$f_0(t) = g(t - 2\alpha) = \frac{A}{6}t^3 - \frac{3}{\alpha^2}t^2 + \frac{9}{2\alpha}t - 1 \qquad \text{on} \quad [0, \beta],$$

we have that

$$f'_0(t) = \frac{A}{2}t^2 - 6\left(\sqrt[3]{\frac{A}{3}}\right)^2 t + \frac{9}{2}\left(\sqrt[3]{\frac{A}{3}}\right) \qquad \text{on} \quad [0, \beta]$$

and that

$$f''_0(t) = At - 6\left(\sqrt[3]{\frac{A}{3}}\right)^2 \qquad \text{on} \quad [0, \beta].$$

Therefore, from the definition of f_0, (c) follows at once. Further we have

$$f''_0(\alpha) < f''_0(2\alpha) = 0,$$

which proves (d) when $A \geqslant 81$. If $0 \leqslant A < 81$, then immediately

$$f'_0(t) = \frac{A}{2}t^2 - \left(\frac{4}{\alpha^2} + \frac{2A}{3}\alpha\right)t + \left(\frac{4}{\alpha} + \frac{A}{6}\alpha^2\right) \qquad \text{on} \quad I$$

and

$$f''_0(t) = At - \left(\frac{4}{\alpha^2} + \frac{2A}{3}\alpha\right) \qquad \text{on} \quad I.$$

Thus (b) follows, similarly. Further we have

$$f''_0(\alpha) \leqslant f''_0(2\alpha) < 0,$$

because $f_0'(2\alpha) < 0$, as easily seen from the definition of f_0. This proves (d) when $0 \leqslant A < 81$, and the proof of (d) is completed.

Now we shall show below that if $\|f\|_I \leqslant 1$ and $\|f^{(3)}\|_I \leqslant A$ then $\|f'\|_I \leqslant \|f_0'\|_I$ and $\|f''\|_I \leqslant \|f_0''\|_I$. Since the argument is somewhat long, we divide it into several steps.

Step I. If $\|f\|_I \leqslant 1$, $\|f^{(3)}\|_I \leqslant A$ and $\|f'\|_I = \|f_0'\|_I$, then f is essentially unique on $[0, \beta^*]$; more precisely,

$$f(t) = \pm f_0(t) \quad \text{or} \quad f(1-t) = \pm f_0(t) \qquad \text{for all} \quad t \in [0, \beta^*].$$

To see this, choose $t_0 \in [0, 1]$ such that $|f'(t_0)| = \|f'\|_I$. Here we may assume without loss of generality that

$$0 \leqslant t_0 \leqslant \tfrac{1}{2} \qquad \text{and} \qquad f'(t_0) = \|f'\|_I.$$

Suppose $t_0 \neq 0$. Then we have $f''(t_0) = 0$, because f'' is continuous and

$$0 \leqslant f'(t_0) - f'(t) = \int_t^{t_0} f''(s)\, ds \qquad \text{for all} \quad 0 \leqslant t \leqslant 1.$$

Since $\|f^{(3)}\|_I \leqslant A$, we then have by (d) that

$$f''(t_0 + s) > f_0''(\alpha - s) \qquad \text{for all} \quad s \in [0, \alpha],$$

and hence that

$$\int_0^t f''(t_0 + s)\, ds > \int_0^t f_0''(\alpha - s)\, ds \geqslant \int_0^t f_0''(s)\, ds$$

for all $t \in (0, \alpha]$.

Therefore

$$f'(t_0 + t) - f_0'(t) = \int_0^t [f''(t_0 + s) - f_0''(s)]\, ds > 0$$

for all $t \in (0, \alpha]$, because $f'(t_0) = \|f'\|_I = \|f_0'\|_I = f_0'(0)$ by hypothesis. But this implies

$$\begin{aligned} f(t_0 + \alpha) - f(t_0) &= \int_0^\alpha f'(t_0 + t)\, dt > \int_0^\alpha f_0'(t)\, dt \\ &= f_0(\alpha) - f_0(0) = 2, \end{aligned}$$

which is a contradiction because $\|f\|_I \leqslant 1$. Hence we conclude that $t_0 = 0$, i.e., $f'(0) = \|f_0'\|_I$ $(= f_0'(0))$.

Suppose $f''(0) < f_0''(0)$. Then, since $\|f^{(3)}\|_I \leqslant A$, we may again apply (d) to infer that $f''(t) < f_0''(t)$ for all $t \in [0, \beta]$. So we obtain

$$f'(t) = f'(0) + \int_0^t f''(s)\, ds < f_0'(0) + \int_0^t f_0''(s)\, ds = f_0'(t)$$

for all $t \in (0, \beta]$, and consequently

$$\begin{aligned} f(\beta) - f(\alpha) &= \int_\alpha^\beta f'(t)\, dt < \int_\alpha^\beta f_0'(t)\, dt \\ &= f_0(\beta) - f_0(\alpha) = -2. \end{aligned}$$

But this is a contradiction.

Suppose $f''(0) > f_0''(0)$. Then, since $f(0) \geqslant -1 = f_0(0)$, $f'(0) = f_0'(0)$ and $f_0(\alpha) = 1$, we can choose $p \in (0, \alpha]$ such that

$$f(p) = f_0(p) \quad \text{and} \quad f(t) > f_0(t) \qquad \text{on} \quad (0, p).$$

Then it follows that $f'(q) < f_0'(q)$ for some $0 < q < p$. From this and the fact that $f'(0) = f_0'(0)$, we observe that $f''(r) < f_0''(r)$ for some $0 < r < q$. Since $\|f^{(3)}\|_I \leqslant A$, we have by (d) that

$$f''(t) < f_0''(t) \qquad \text{on} \quad [r, \beta].$$

Hence

$$f'(t) = f'(q) + \int_q^t f''(s)\, ds < f_0'(q) + \int_q^t f_0''(s)\, ds = f_0'(t)$$

for all $t \in [q, \beta]$, which implies $f(\beta) - f(\alpha) < f_0(\beta) - f_0(\alpha) = -2$, a contradiction.

Suppose $f''(0) = f_0''(0)$. Then $f''(t) \leqslant f_0''(t)$ on $[0, \beta]$, and hence $f'(t) \leqslant f_0'(t)$ on $[0, \beta]$ because $f'(0) = f_0'(0)$. It follows that $f(t) \leqslant f_0(t)$ on $[\alpha, \beta]$, $f(\beta) = -1$ and $f(\alpha) = 1$. Therefore $f'(t) = f_0'(t)$ on $[\alpha, \beta]$, and consequently $f(t) = f_0(t)$ on $[\alpha, \beta]$. By a similar argument we see that $f(t) = f_0(t)$ on $[0, \alpha]$, and further that $f(t) = f_0(t)$ on $[0, \beta^*]$.

Step II. If $\|f\|_I \leqslant 1$ and $\|f^{(3)}\|_I \leqslant A$, then $\|f'\|_I \leqslant \|f_0'\|_I$.

In fact, if we should assume the contrary: $c\|f'\|_I = \|f_0'\|_I$ for some $0 < c < 1$, then, since $\|cf\|_I \leqslant 1$ and $\|cf^{(3)}\|_I \leqslant A$, we should have by Step I that $\|cf\|_I \geqslant \|f_0\|_I = 1$, or equivalently that $\|f\|_I \geqslant 1/c > 1$. But this is a contradiction because $\|f\|_I \leqslant 1$ by hypothesis.

Step III. If $\|f\|_I \leqslant 1$, $\|f^{(3)}\|_I \leqslant A$ and $\|f''\|_I = \|f_0''\|_I$, then $\|f'\|_I = \|f_0'\|_I$.

To see this, choose $t_0 \in I$ such that $\|f''\|_I = |f''(t_0)|$. Here we may assume without loss of generality that

$$0 \leqslant t_0 \leqslant \tfrac{1}{2} \qquad \text{and} \qquad f''(t_0) = -\|f''\|_I.$$

First suppose $0 \leqslant t_0 < \alpha$. Since $\|f^{(3)}\|_I \leqslant A$, we then have by (d) that

$$f''(t) \leqslant f_0''(t_0 - t) \qquad \text{for all} \quad t \in [0, t_0]$$

and that

$$f''(t) \leqslant f_0''(t) \qquad \text{for all} \quad t \in [t_0, \beta].$$

It follows that for all $t \in [\alpha, \beta]$

$$f'(t) - f'(0) = \int_0^t f''(s)\, ds \leqslant \int_0^t f_0''(s)\, ds = f_0'(t) - f_0'(0).$$

Therefore, if $f'(0) < f_0'(0)$, then we have $f'(t) < f_0'(t)$ for all $t \in [\alpha, \beta]$; consequently

$$\begin{aligned} f(\beta) - f(\alpha) &= \int_\alpha^\beta f'(t)\, dt < \int_\alpha^\beta f_0'(t)\, dt \\ &= f_0(\beta) - f_0(\alpha) = -2, \end{aligned}$$

which is a contradiction. Thus it follows that $f'(0) \geqslant f_0'(0) = \|f_0'\|_I$. Hence, by Step II, $\|f'\|_I = \|f_0'\|_I$.

Next suppose $\alpha \leqslant t_0 \leqslant \frac{1}{2}$. By a similar argument we have that for every $t \in (0, \alpha)$

$$\begin{aligned} f'(t_0) - f'(t_0 - t) &= \int_{t_0 - t}^{t_0} f''(s)\, ds \leqslant \int_0^t f_0''(s)\, ds \\ &< \int_{\alpha - t}^{\alpha} f_0''(s)\, ds = f_0'(\alpha) - f_0'(\alpha - t). \end{aligned}$$

Since (a) implies $f_0'(\alpha) = 0$, it follows that $f'(t_0 - t) > f'(t_0) + f_0'(\alpha - t)$ on $(0, \alpha)$. Thus if $f'(t_0) \geqslant 0$ then it follows that $f'(t_0 - t) > f_0'(\alpha - t)$ on $(0, \alpha)$, and hence we get

$$f(t_0) - f(t_0 - \alpha) = \int_0^\alpha f'(t_0 - t)\, dt > \int_0^\alpha f_0'(\alpha - t)\, dt = 2,$$

which is a contradiction. On the other hand, if $f'(t_0) < 0$ then we have, for every $t \in [0, \alpha]$,

$$f'(t_0 + t) = f'(t_0) + \int_{t_0}^{t_0+t} f''(s)\, ds$$

$$< \int_{\alpha - t}^{\alpha} f_0''(s)\, ds = -f_0'(\alpha - t).$$

Therefore

$$f(t_0 + \alpha) - f(t_0) = \int_0^{\alpha} f'(t_0 + t)\, dt$$

$$< \int_0^{\alpha} -f_0'(\alpha - t)\, dt = -2,$$

which is a contradiction, too.

Step IV. If $\|f\|_I \leqslant 1$ and $\|f^{(3)}\|_I \leqslant A$, then $\|f''\|_I \leqslant \|f_0''\|_I$.

This is now an immediate consequence, and the proof of the theorem is completed.

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